The Invariance of the Scalar Decay Law

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We consider a turbulent mixing device with two or many inlets. The geometries of the inlets need not be the same and the flow rates through the inlets need not be the same. We take some arbitrary fraction of the inlets and add tracer at a uniform composition C_o to the streams entering through these inlets, labeled 1 inlets. The root-mean-square concentration fluctuation in the mixer is now measured at all points giving the scalar decay law.

$$\Gamma = \sqrt{\overline{C'^2}} = \Gamma \text{ (Position)} \tag{1}$$

C is the tracer composition at any time and position and C' is the corresponding fluctuation about the mean at that position.

The question to be considered is what is the value of Γ if the tracer is added not to the 1 inlets, but to the remaining fraction of the inlets, labeled 2 inlets.

ANALYSIS

When the tracer is added to the 1 inlets call it species A and when added to the 2 inlets call it species B. Then the instantaneous concentration fields are described by the normal convective diffusion law for dilute systems.

$$\mathcal{L}(C_A) = \mathcal{L}(C_B) = 0 \tag{2}$$

where $\mathcal{L}$ is the operator

$$\mathcal{L} \equiv \left(\frac{\partial}{\partial\theta} + \underline{V} \cdot \nabla - \mathcal{D}\nabla^2\right) \tag{3}$$

$\underline{V}$ is the instantaneous velocity, and $\mathcal{D}$ is the molecular diffusion coefficient of tracer in the solvent. The operator $\mathcal{L}$ is the same in both cases since the velocity and diffusivity are the same. The boundary conditions on all solid boundaries are

$$\frac{\partial C_A}{\partial n} = \frac{\partial C_B}{\partial n} = 0 \tag{3a}$$

where n is the derivative normal to the boundary.

At the 1 inlets

$$C_A = C_{Ao} \tag{3b}$$

$$C_B = 0 \tag{3c}$$

At the 2 inlets

$$C_A = 0 \tag{3d}$$

$$C_B = C_{Bo} \tag{3e}$$

The above boundary conditions assume impervious walls and no back diffusion into the inlets, hardly restrictive assumptions.

We now define

$$\frac{C_A - C_{A1}}{C_{A2} - C_{A1}} = f_A \tag{4}$$

$$\frac{C_B - C_{B1}}{C_{B2} - C_{B1}} = f_B \tag{5}$$

where subscript 1 refers to the concentration in inlets 1 and 2 to the concentration in inlets 2. It now follows from Equations (4) and (2), and (5) and (2) that

$$\mathcal{L}(f_A) = \mathcal{L}(f_B) = 0 \tag{6}$$

From Equations (4) and (5)

At the 1 inlets

$$f_A = f_B = 0 \tag{7}$$

At the 2 inlets

$$f_A = f_B = 1 \tag{8}$$

And from Equation (3a)

At the solid boundaries

$$\frac{\partial f_A}{\partial n} = \frac{\partial f_B}{\partial n} = 0 \tag{9}$$

Thus Equations (6), (7), (8), and (9) show that

$$f_A = f_B \tag{10}$$

everywhere since the equations and boundary conditions are identical. From Equations (10) and (3b) to (3e) we obtain

$$\frac{C_A}{C_{Ao}} = 1 - \frac{C_B}{C_{Bo}} \tag{11}$$

And this holds instantaneously at every position and time irrespective of the velocity field or geometry (assuming that the velocity fields are the same in both cases). Time averaging yields the equation

$$\frac{\overline{C}_A}{C_{Ao}} = 1 - \frac{\overline{C}_B}{C_{Bo}} \tag{12}$$

which holds at every point. Defining the concentration fluctuation as usual, we get

$$\frac{(\overline{C}_A + C_A')}{C_{Ao}} = 1 - \frac{(\overline{C}_B + C_B')}{C_{Bo}} \tag{13}$$

and equating Equations (12) and (13) and combining, we get

$$\Gamma_A = \frac{C_{Ao}}{C_{Bo}} \Gamma_B \tag{14}$$

So if the tracer concentrations used in the 1 and 2 inlets are the same

$$\Gamma_A = \Gamma_B = \Gamma \tag{15}$$

and the decay law is the same whether we add the tracer to the 1 inlets or the 2 inlets.

If now there is a third set of inlets which contains no tracer in either case so that at the 3 inlets $C_A = C_B = 0$ always, then the above results cannot hold, since in order for the boundary conditions on f_A to equal those on f_B it is necessary that at the 3 inlets

$$\frac{C_{A1}}{C_{A2} - C_{A1}} = \frac{C_{B1}}{C_{A2} - C_{A1}}$$

an impossible condition. Hence the decay law will depend upon which set of inlets contains the tracer.

The results of this note also have implications concerning the problem of chemical reactions with mixing which will be considered later.

ACKNOWLEDGMENT

The author is grateful to the National Science Foundation for support of this work and to D. W. Condiff for useful discussions.